Exact Resuslts on Massless $\mathbb{Z}_3$-QCD via Anomaly Matching

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References: 1710.08923[hep-th], 1711.10487[hep-th]

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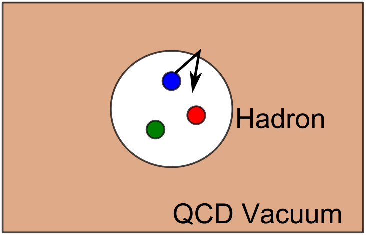
Introduction: Confinement and chiral symmetry, $\mathbb{Z}_3$-QCD

Confinement ⇒ Chiral symmetry breaking?

Heuristic argument (like the bag model) indicates that

Confinement ⇒ Chiral symmetry breaking



Can we make the EXACT statement related to this problem based on QCD?

Confinement and Wilson loop

To define confinement, we first consider pure Yang-Mills.

Put static quarks separated by L, then they are confined due to the color string:

If this sting can break, then quarks are deconfined. (Wilson, 1974)

$$\exp(-TV(L)) = \begin{cases} \exp(-\sigma T L) & \text{confinement} \\ 1 & \text{deconfinement} \end{cases}$$

The operator defining this quark potential is called Wilson lines:

$$W(C) = \mathrm{tr}\left[\mathcal{P}\exp\left(\mathrm{i}\int_C a\right)\right].$$

Confinement and Center symmetry

Confinement/deconfinement is characterized by *center symmetry*.

$SU(N)$ gauge field a on a spacetime manifold M
$=$ patches $\{U_i\}$, one-forms $a^{(i)}$ on U_i, and transition functions $g_{ij}\in SU(N)$ on $U_i\cap U_j$, satisfying

$$a^{(j)}=g_{ij}^{-1}a^{(i)}g_{ij}+g_{ij}^{-1}\mathrm{d}g_{ij},\ g_{ij}g_{jk}g_{ki}=1.$$

Center symmetry is the symmetry to change transition functions by centers of $SU(N)$:

$$g_{ij}\mapsto g_{ij}\exp\left(\frac{2\pi\mathrm{i}}{N}n_{ij}\right)\equiv\omega^{n_{ij}}g_{ij}.$$

The order parameter is Wilson loop: $W(C)\mapsto \omega W(C)$.

(This is one-form symmetry: Gaiotto, Kapustin, Seiberg, Willet (2014))

Center symmetry and fundamental quarks

Center symmetry is explicitly broken with fundamental quarks
$\Rightarrow$ Let's consider a nice setup ($\mathbb{Z}_{N_C}$-QCD (Kouno et al. 2012, ...)).

Prepare $N_F = N_C$ quarks, and put the boundary condition on the quark field as $(f = 1, \dots, N)$

$$q_f(\boldsymbol{x}, x_4 + L) = \omega^f q_f(\boldsymbol{x}, x_4).$$

This theory has the $(\mathbb{Z}_N)_{\mathrm{shift,center}}$ symmetry, defined by

$$\Phi = \mathrm{tr}_{\mathrm{c}} \left[P \exp \left(\mathrm{i} \int_{S^1} a \right) \right] \mapsto \omega\, \Phi,\ q_f \mapsto q_{f+1}.$$

$\mathbb{Z}_N$-QCD

Using operators of QCD, the partition function of $\mathbb{Z}_N$-QCD is

$$\mathcal{Z}_{\mathbb{Z}_N-\mathrm{QCD}}=\mathrm{tr}_{\mathcal{H}}\left[\mathrm{e}^{-L(\widehat{H}-\mu\widehat{Q})}\exp\left(\mathrm{i}\sum_{f=1}^{N}\frac{2\pi f}{N}\widehat{Q}_f\right)\right].$$

We will see that anomaly matching argument constrains the possible phases of $\mathcal{Z}_{\mathbb{Z}_N-\mathrm{QCD}}$:

(Discrete) chiral symmetry is broken in the center-symmetric phase, e.g.

$$T_{\mathrm{deconf}}\le T_{\mathrm{chiral}}$$

for massless $\mathbb{Z}_N$-QCD. (1710.08923, 1711.10487)

(Similar, related results for pure YM, Bifundamental QCD, adjoint QCD, etc.:
Gaiotto, Kapustin, Komargodski, Seiberg (1703); Tanizaki, Kikuchi (1705);
Komargodski, Sulejmanpasic, Unsal (1706); Shimizu, Yonekura (1706); ...)

Anomaly matching

't Hooft anomaly $\equiv$ Global symmetry that cannot be gauged.

Theorem

't Hooft anomaly is renormalization-group invariant. *('t Hooft, '80)*

Classic Example

Massless QCD has symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ with an 't Hooft anomaly:

$$D^\mu J^a_\mu = 0, \quad D^\mu J^{5a}_\mu = \frac{1}{4\pi^2}\mathrm{tr}\left(T^a\left\{F_V^2 + \frac{1}{3}F_A^2 + \cdots\right\}\right)$$

Conservation of J^{5a} is violated by background flavor gauge fields.
$\Rightarrow$ This is matched by ChSB and the WZW term for massless pions.

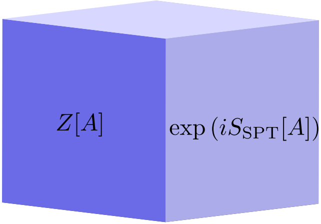
To match the anomaly, ground states must be *nontrivial*.

“Proof” of anomaly matching

A: Background gage field for global symmetry G
$A\mapsto A+\mathrm{d}_A\theta$: G-gauge transformation, but this has an anomaly:

$$Z[A+\mathrm{d}_A\theta]=Z[A]\exp(\mathrm{i}\mathcal{A}[A,\theta]).$$

Combined system with (d+1)D SPT phase is G-gauge invariant.



$\Rightarrow$ Our system (or, surface states) must be nontrivial.

Technique: New 't Hooft anomaly of massless 3-flavor QCD

Symmetry of massless QCD

QCD Lagrangian:

$$S = \frac{1}{2g^2}\int \mathrm{tr}(G \wedge *G) + \int \sum_{f=1}^{N_F} \overline{q}_f \gamma_\mu D_\mu q.$$

Symmetry of massless QCD (i.e. Lagrangian + $\mathcal{D}\overline{q}\mathcal{D}q$):

$$\frac{SU(N_F)_L \times SU(N_F)_R \times U(1)_V \times (\mathbb{Z}_{2N_F})_A}{\mathbb{Z}_{N_C} \times (\mathbb{Z}_{N_F})_L \times (\mathbb{Z}_{N_F})_R \times \mathbb{Z}_2}$$

- For later purpose, I evade the double counting correctly (e.g. $U(N) = [SU(N) \times U(1)]/\mathbb{Z}_N$, $U(1)_L \times U(1)_R = [U(1)_V \times U(1)_A]/\mathbb{Z}_2$)
- Due to the gauge invariance, symmetry must be divided by $\mathbb{Z}_{N_C} \subset SU(N_C)$.

Further comments on chiral symmetry

One way to think about $(\mathbb{Z}_{2N_F})_A$ is that it is anomaly-free subgroup of $U(1)_A$.

For later application, let us see another perspective:

$$(\mathbb{Z}_{2N_F})_A \subset SU(N_F)_L \times SU(N_F)_R \times U(1)_V .$$

Indeed, the generator of $(\mathbb{Z}_{2N_F})_A$ can be written as

$$\mathrm{e}^{\frac{2\pi}{2N_F}\mathrm{i}\gamma_5}\mathbf{1}_{N_F} = \exp\left(\frac{2\pi}{N_F}\mathrm{i}\frac{1+\gamma_5}{2}\mathrm{diag}[1,\dots,1,1-N_F]\right)\mathrm{e}^{-2\pi\mathrm{i}/(2N)}$$

Quick message: SSB of $(\mathbb{Z}_{2N_F})_A$ implies continuous ChSB.

Gauging vector-like symmetry: Part 1

We pay attention to the vector-like symmetry,

$$\frac{SU(N_F)_V \times U(1)_V}{(\mathbb{Z}_{N_C}) \times (\mathbb{Z}_{N_F})},$$

and $(\mathbb{Z}_{2N})_{\mathrm{axial}}$.

To detect the 't Hooft anomaly, we introduce the gauge fields for vector-like symmetry (1710.08923, 1711.10487) (ref. Kapustin, Seiberg (2014));

- $SU(N_F)_V$ one-form gauge field: A_f
- $U(1)_V$ one-form gauge field: A_q
- $(\mathbb{Z}_{N_C})$ two-form gauge field: B_c
- $(\mathbb{Z}_{N_F})$ two-form gauge field: B_f

Gauging vector-like symmetry: Part 2

Question What is the meaning of two-form gauge fields?

Connection formula on double overlaps of patches $U_i \cap U_j$
($g_{ij}^{(c)} \in SU(N_c), g_{ij}^{(f)} \in SU(N_f), g_{ij}^{(q)} \in U(1)_V$):
For gauge field,

$$\begin{aligned} a^{(j)} &= g_{ij}^{(c)}{}^{-1} a^{(i)} g_{ij}^{(c)} + g_{ij}^{(c)}{}^{-1} \mathrm{d} g_{ij}^{(c)}, \\ A_f^{(j)} &= g_{ij}^{(f)}{}^{-1} A_f^{(i)} g_{ij}^{(f)} + g_{ij}^{(f)}{}^{-1} \mathrm{d} g_{ij}^{(f)}, \\ A_q^{(j)} &= g_{ij}^{(q)}{}^{-1} A_q^{(i)} g_{ij}^{(q)} + g_{ij}^{(q)}{}^{-1} \mathrm{d} g_{ij}^{(q)}. \end{aligned}$$

For quark field,

$$q^{(j)} = \left(g_{ij}^{(c)} \otimes g_{ij}^{(f)} \otimes g_{ij}^{(q)} \right) q^{(i)}.$$

Gauging vector-like symmetry: Part 3

Consistency requires the cocycle condition for $\left(g_{ij}^{(c)} \otimes g_{ij}^{(f)} \otimes g_{ij}^{(q)}\right)$, but not for each of them:

$$g_{ij}^{(c)} g_{jk}^{(c)} g_{ki}^{(c)} = \exp\left(\frac{2\pi \mathrm{i}}{N_C} n_{ij}^{(c)}\right) \in \mathbb{Z}_{N_C},$$

$$g_{ij}^{(f)} g_{jk}^{(f)} g_{ki}^{(f)} = \exp\left(\frac{2\pi \mathrm{i}}{N_F} n_{ij}^{(f)}\right) \in \mathbb{Z}_{N_F},$$

$$g_{ij}^{(q)} g_{jk}^{(q)} g_{ki}^{(q)} = \exp\left(-\frac{2\pi \mathrm{i}}{N_C} n_{ij}^{(c)} - \frac{2\pi \mathrm{i}}{N_F} n_{ij}^{(f)}\right) \in U(1)_V.$$

$[\{(n_{ij}^{(c)}, U_i \cap U_j)\}_{ij}]$ and $[\{(n_{ij}^{(f)}, U_i \cap U_j)\}_{ij}]$ are characterized by

$$B_c \in H^2(M, \mathbb{Z}_{N_C}),\ B_f \in H^2(M, \mathbb{Z}_{N_F}).$$

Mixed 't Hooft anomaly for massless QCD

$(\mathbb{Z}_{2N})_{\mathrm{axial}}$ can change $\mathcal{Z}$ as

$$\mathcal{Z}[(A_{\mathrm{f}},A_{\mathrm{q}},B_{\mathrm{c}},B_{\mathrm{f}})]\mapsto \mathcal{Z}[(A_{\mathrm{f}},A_{\mathrm{q}},B_{\mathrm{c}},B_{\mathrm{f}})]\exp(\mathrm{i}\mathcal{A}).$$

The 't Hooft anomaly $\mathcal{A}$ is given by Fujikawa method:

$$\mathcal{A}=2\frac{2\pi}{2N}\frac{1}{8\pi^2}\int \mathrm{tr}_{\mathrm{c,f}}\left[F\wedge F\right],$$

F: field strength of $[SU(N_C)\times SU(N_F)\times U(1)_V]/[\mathbb{Z}_{N_C}\times \mathbb{Z}_{N_F}]$.

For simplicity of expression, let us set $N_C=N_F=N$, then

$$\mathcal{A}=-\frac{N}{2\pi}\int B_{\mathrm{c}}\wedge B_{\mathrm{f}}\in \frac{2\pi}{N}\mathbb{Z}.$$

After gauging the vector symmetry correctly, $(\mathbb{Z}_{2N})_A$ is broken by $\mathcal{A}$.

Summary of computation

We show that massless N-flavor QCD has a mixed 't Hooft anomaly between

$$\frac{SU(N_F)_V\times U(1)_V}{(\mathbb{Z}_{N_C})\times(\mathbb{Z}_{N_F})}\quad \text{and} \quad (\mathbb{Z}_{2N})_{\mathrm{axial}}.$$

The anomaly is given as $(N_C=N_F=N)$

$$\mathcal{Z}[(A_\mathrm{f},A_\mathrm{q},B_\mathrm{c},B_\mathrm{f})]\mapsto \mathcal{Z}[(A_\mathrm{f},A_\mathrm{q},B_\mathrm{c},B_\mathrm{f})]\exp\left(-\frac{\mathrm{i}N}{2\pi}\int B_\mathrm{c}\wedge B_\mathrm{f}\right).$$

We will discuss consequences of this anomaly.

Application: Constraint on the phase diagram of massless $\mathbb{Z}_3$-QCD

Anomaly and S^1 compactification

Remark: 't Hooft anomaly does not constrain S^1-compactified QFTs.

Example: Consider 3D Dirac fermion, $\overline{\psi}\sigma_i\partial_i\psi$. Symmetry $= U(1)\times$ T

$$\mathsf{T} : Z[A] \mapsto Z[A] \exp\left(\frac{\mathrm{i}}{4\pi}\int A\mathrm{d}A\right).$$

At finite temperatures, all the fermions get KK mass, $\frac{2\pi}{\beta}(n+1/2)$; the system is gapped and has the unique ground state.

Theorem

Despite this fact, for massless $\mathbb{Z}_N$-QCD, the new anomaly $\mathcal{A} = -\frac{N}{2\pi}\int B_\mathrm{c}\wedge B_\mathrm{f}$ of massless QCD still gives constraints.(1710.08923)

Symmetry and Anomaly of massless $\mathbb{Z}_N$-QCD

There are three symmetries for massless $\mathbb{Z}_N$-QCD as a 3D QFT:

$$(\mathbb{Z}_N)_{\mathrm{shift,center}}, \quad \frac{U(1)_F^{N-1}\times U(1)_V}{(\mathbb{Z}_N)_C\times(\mathbb{Z}_N)_F}, \quad \text{and} \quad (\mathbb{Z}_{2N})_{\mathrm{axial}}$$

We can gauge $(\mathbb{Z}_N)_{\mathrm{shift}}$ and $[U(1)_F^{N-1}\times U(1)_V]/[(\mathbb{Z}_N)_C\times(\mathbb{Z}_N)_F]$ by introducing the following two-form gauge fields (+ ordinary ones):

$$B_{\mathrm{c}} = B_{\mathrm{c}}^{(1)}\wedge L^{-1}\mathrm{d}x^4 + B_{\mathrm{c}}^{(2)}, \quad B_{\mathrm{f}} = B_{\mathrm{f}}^{(2)}.$$

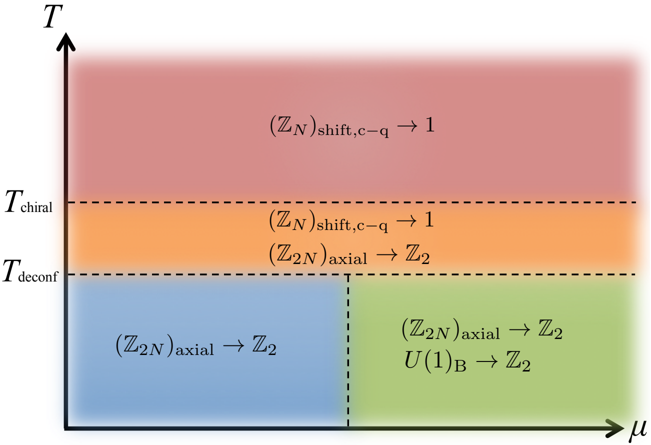
Substituting it into the four-dimensional anomaly, we obtain the anomaly for massless $\mathbb{Z}_N$-QCD:

$$\mathcal{A} = -\frac{N}{2\pi}\int B_{\mathrm{c}}^{(1)}\wedge B_{\mathrm{f}}^{(2)} \in \frac{2\pi}{N}\mathbb{Z}.$$

(1710.08923, 1711.10487)

One of possible phase diagrams

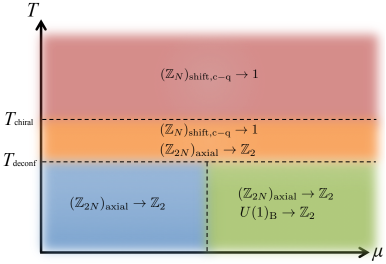
Anomaly matching says that the phase is nontrivial at any T and μ:


T
$(\mathbb{Z}_N)_{\mathrm{shift,c-q}} \to 1$
T_{chiral}
$(\mathbb{Z}_N)_{\mathrm{shift,c-q}} \to 1$
$(\mathbb{Z}_{2N})_{\mathrm{axial}} \to \mathbb{Z}_2$
T_{deconf}
$(\mathbb{Z}_{2N})_{\mathrm{axial}} \to \mathbb{Z}_2$
$(\mathbb{Z}_{2N})_{\mathrm{axial}} \to \mathbb{Z}_2$
$U(1)_{\mathrm{B}} \to \mathbb{Z}_2$
μ

(1711.10487)

Comment on Possible phase diagrams

We have a mixed 't Hooft anomaly between $(\mathbb{Z}_N)_{\text{shift}}$, $\frac{U(1)_F^{N-1}\times U(1)_V}{(\mathbb{Z}_N)_C\times(\mathbb{Z}_N)_F}$, and $(\mathbb{Z}_{2N})_{\text{axial}}$.



T
$(\mathbb{Z}_N)_{\text{shift,c-q}}\to 1$
T_{chiral}
$(\mathbb{Z}_N)_{\text{shift,c-q}}\to 1$
$(\mathbb{Z}_{2N})_{\text{axial}}\to\mathbb{Z}_2$
T_{deconf}
$(\mathbb{Z}_{2N})_{\text{axial}}\to\mathbb{Z}_2$
$(\mathbb{Z}_{2N})_{\text{axial}}\to\mathbb{Z}_2$
$U(1)_{\text{B}}\to\mathbb{Z}_2$
μ

1. System is conformal or breaks some of symmetries
2. When anomaly is matched by SSB,

$$T_{\text{deconf}}(\mu)\le T_{\text{chiral}}(\mu)$$

if flavor is unbroken.

Note: These constraints are not necessarily satisfied by PNJL model. Ginzburg-Landau description does not respect anomaly matching, so we must go beyond!

Summary

- We find a new 't Hooft anomaly of massless QCD.
  Anomaly matching gives nonperturbative constraints on low-energy physics.
- Application to massless $\mathbb{Z}_N$-QCD:
  The phase is always non-trivial at any T and μ.
- If the vector-like flavor is unbroken, $T_{\mathrm{deconf}} \le T_{\mathrm{chiral}}$;

Confinement ⇒ Chiral symmetry breaking